ON THE DISTRIBUTIVITY EQUATION FOR UNI-NULLNORMS

YA-MING WANG AND HUA-WEN LIU

A uni-nullnorm is a special case of 2-uninorms obtained by letting a uninorm and a nullnorm share the same underlying t-conorm. This paper is mainly devoted to solving the distributivity equation between uni-nullnorms with continuous Archimedean underlying t-norms and t-conorms and some binary operators, such as, continuous t-norms, continuous t-conorms, uninorms, and nullnorms. The new results differ from the previous ones about the distributivity in the class of 2-uninorms, which have not yet been fully characterized.

Keywords: fuzzy connectives, uni-nullnorms, T-norms, T-conorms, nullnorms, uninorms, distributivity equation

Classification: 46F10, 62E86

1. INTRODUCTION

Uninorms as aggregation operators generalizing and unifying the concepts of t-norms and t-conorms were introduced by Yager and Rybalov [30], and have been proved to be useful in many fields like fuzzy system modeling, decision making and so on [31]. Fodor et al. [7] studied the structure of uninorms and gave the characterization of uninorms, which can be built up from t-norms and t-conorms by using an ordinal sum structure. And Fodor and Li et al. gave a single-point characterization of uninorms in recent years [8, 10, 11]. Nullnorms and t-operators as another generalizations of t-norms and t-conorms were respectively introduced by Calvo [3] and Mas et al. [12], and they have been proved to be equivalent [13]. 2-uninorms as a generalization of uninorms and nullnorms, and such generalization further extended to n-uninorms, were introduced by Akella in [2]. However, the structure of 2-uninorms has not yet been fully characterized. Recently, Sun et al. [26] developed the concept of a uni-nullnorm (null-uninorm) by letting a uninorm and a nullnorm share the same underlying t-conorm (t-norm). Uni-nullnorms and null-uninorms as special cases of 2-uninorms generalize uninorms and nullnorms and have been proved that they are a pair of dual binary operations. After that, Sun et al. showed the full characterizations of uni-nullnorms with continuous Archimedean underlying t-norms and t-conorms [27]. Although a uni-nullnorm with continuous Archimedean underlying t-norms and a t-conorm is a special...
case of 2-uninorms, it is not included in the structures of 2-uninorms given in [5] that just have five special structures of 2-uninorms.

The role of functional equations involving aggregation operators is very important in theories of fuzzy sets and fuzzy logic. According to the literature we can find, there are a lot of studies on distributivity equation about uninorms, nullnorms, semi-uninorms, semi-nullnorms and semi-t-operators [11, 13, 16-20, 22-25, 29]. But for 2-uninorms, there are only a few studies on distributivity equation [5, 15, 28]. The reason for the lack of study on the distributivity equation about 2-uninorms is that there is no complete characterization of 2-uninorms. Now Sun et al. have given the full description of uni-nullnorms, which is a special case of 2-uninorms. Obviously, the study of distributivity about uni-nullnorms is the preparation for the discussion of distributive 2-uninorms, so it is necessary to characterize the distributivity equation of uni-nullnorms.

This paper is organized as follows. In Section 2 we present the definitions and structures of some concerning binary operators used later in the paper. In Section 3 we investigate the distributivity between uni-nullnorms and continuous t-norms (t-conorms). After that, we characterize the distributivity equation between uni-nullnorms and uninorms in Section 4, and the distributivity equation between uni-nullnorms and nullnorms in Section 5. Section 6 is conclusion and further work.

2. PRELIMINARIES

We only recall some facts on t-norms, t-conorms, uninorms, nullnorms and uni-nullnorms, which will be used throughout the paper. More details about them can be found in [3, 4, 7, 9, 22, 27].

Definition 2.1. (Klement et al. [9]) A t-norm is a binary function \( T : [0,1]^2 \rightarrow [0,1] \) which is a commutative, associative, increasing binary operator with a neutral element 1. A t-conorm is a binary function \( S : [0,1]^2 \rightarrow [0,1] \) which is a commutative, associative, increasing binary operator with a neutral element 0. A t-norm \( T \) (t-conorm \( S \)) is called Archimedean if for each \((x, y) \in (0,1)^2\) there is an \( n \in \mathbb{N} \) such that \( T(x, \cdots, x) = x^n_T < y \) \((x^n_S > y)\).

A t-norm and a t-conorm are a pair of dual operators, we can obtain the corresponding results for t-conorms by interchanging the words t-norms and t-conorms and the roles of 0 and 1, respectively. Thus we only give some definitions and lemmas about t-norms.

For a t-norm \( T \), an element \( c \in (0,1) \) is called a nilpotent element of \( T \) if there exists some \( n \in \mathbb{N} \) such that \( c^{(n)}_T = 0 \). \( T \) is said to be strictly monotone (SM) if \( T(x, y) < T(x, z) \) whenever \( x > 0 \) and \( y < z \). \( T \) satisfies the cancellation law (CL) if \( T(x, y) = T(x, z) \) implies \( x = 0 \) or \( y = z \). \( T \) satisfies the conditional cancellation law (CCL) if \( T(x, y) = T(x, z) > 0 \) implies \( y = z \). For a continuous t-norm \( T \), \( T \) is called strict if it is strictly monotone, and \( T \) is called nilpotent if every \( c \in (0,1) \) is a nilpotent element of \( T \). Any continuous Archimedean t-norms is either strict or nilpotent.

Lemma 2.2. (Klement et al. [9]) A continuous t-norm \( T \) is Archimedean if and only if it satisfies CCL.

Lemma 2.3. (Klement et al. [9]) A continuous t-norm \( T \) is Archimedean if and only if \( T(x, x) < x \) for all \( x \in (0,1) \).
Definition 2.4. (Fodor et al. [7]) A uninorm is a binary function \( U : [0, 1]^2 \rightarrow [0, 1] \) which is commutative, associative, increasing in each variable and there is a neutral element \( f \in [0, 1] \) such that \( U(f, x) = x \) for all \( x \in [0, 1] \).

Evidently, a uninorm with a neutral element \( f = 1 \) is a t-norm and a uninorm with a neutral element \( f = 0 \) is a t-conorm. For any uninorm \( U \), we have \( U(0, 1) \in \{0, 1\} \), and \( U \) is called conjunctive when \( U(0, 1) = 0 \) and disjunctive when \( U(0, 1) = 1 \). By \( U_f \) we denote the family of all uninorms with neutral element \( f \in [0, 1] \). In what follows we give the description of uninorms by using the notation \( A(f) = [0, f) \times (f, 1] \cup (f, 1] \times [0, f) \).

Theorem 2.5. (Fodor et al. [7]) Let \( f \in (0, 1) \). \( U \in U_f \) if and only if

\[
U(x, y) = \begin{cases} 
   fT_U\left(\frac{x}{f}, \frac{y}{f}\right) & \text{if } (x, y) \in [0, f]^2, \\
   f + (1 - f)S_U\left(\frac{x - f}{1 - f}, \frac{y - f}{1 - f}\right) & \text{if } (x, y) \in [f, 1]^2, \\
   C(x, y) & \text{if } (x, y) \in A(f),
\end{cases}
\]  

(1)

where \( T_U \) and \( S_U \) are respectively a t-norm and a t-conorm, the associative and increasing operation \( C : A(f) \rightarrow [0, 1] \) fulfills \( \min(x, y) \leq C(x, y) \leq \max(x, y) \) for \( (x, y) \in A(f) \).

Definition 2.6. (Fodor et al. [7]) Consider \( f \in (0, 1) \). A binary operator \( U : [0, 1]^2 \rightarrow [0, 1] \) is a representable uninorm if and only if there exists a continuous strictly increasing function \( h : [0, 1] \rightarrow [-\infty, +\infty] \) with \( h(0) = -\infty, h(f) = 0 \) and \( h(1) = +\infty \) such that

\[
U(x, y) = h^{-1}(h(x) + h(y))
\]

for all \( (x, y) \in [0, 1] \setminus \{(0, 1), (1, 0)\} \) and \( U(0, 1) = U(1, 0) \in \{0, 1\} \). The function \( h \) is usually called an additive generator of \( U \).

Recall that there are no continuous uninorms with neutral element \( f \in (0, 1) \). In fact, representable uninorms were characterized as those uninorms that are continuous in \( [0, 1]^2 \setminus \{(0, 1), (1, 0)\} \) (see [22]) as well as those that are strictly increasing in the open unit square (see [8, 22]).

Definition 2.7. (Calvo et al. [3]) A nullnorm is a binary function \( V : [0, 1]^2 \rightarrow [0, 1] \) which is commutative, associative, increasing and has a zero element \( k \in [0, 1] \) such that

(i) \( V(0, x) = V(x, 0) = x \) for all \( x \leq k \),

(ii) \( V(1, x) = V(x, 1) = x \) for all \( x \geq k \).

By Definition 2.7 the case \( k = 0 \) leads back to a t-norm, while the case \( k = 1 \) leads back to a t-conorm. By \( V_k \) we denote the family of all nullnorms with the zero element \( k \in [0, 1] \).

Theorem 2.8. (Calvo et al. [3]) Let \( k \in (0, 1) \). \( V \in V_k \) if and only if there exists a t-norm \( T_V \) and a t-conorm \( S_V \) such that

\[
V(x, y) = \begin{cases} 
kS_V\left(\frac{x}{k}, \frac{y}{k}\right) & \text{if } (x, y) \in [0, k]^2, \\
k + (1 - k)T_V\left(\frac{x - k}{1 - k}, \frac{y - k}{1 - k}\right) & \text{if } (x, y) \in [k, 1]^2, \\
k & \text{otherwise}.
\end{cases}
\]

(2)
Definition 2.9. (De Baets [4]) An element $s \in [0, 1]$ is called an idempotent element of operation $F : [0, 1]^2 \to [0, 1]$ if $F(s, s) = s$. Operation $F$ is called idempotent if all elements from $[0, 1]$ are idempotent.

Definition 2.10. (Akella [2], Fechner et al. [6]) Let $G$ be a binary operator on $[0, 1]$ which is commutative. Then $\{e_1, e_2\}_a$ is called the 2-neutral element of $G$ if $G(e_1, x) = x$ for all $x \leq a$ and $G(e_2, x) = x$ for all $x \geq a$ where $0 < a < 1$ and $e_1 \in [0, a]$, $e_2 \in [a, 1]$.

Definition 2.11. (Akella [2], Fechner et al. [6]) A binary operator $G$ on $[0, 1]$ is a 2-uninorm if it is commutative, associative, increasing in both the variable and has a 2-neutral element $\{e_1, e_2\}_a$.

Definition 2.12. (Sun et al. [26]) A uni-nullnorm $G$ is a 2-uninorm having a 2-neutral element $\{e, 1\}_a$ with $a$ being the zero element over $[e, 1]$.

Note that a uni-nullnorm is a special case of a 2-uninorm, where the underlying upper uninorm of the 2-uninorm is a t-norm. Obviously, uni-nullnorms generalize both uninorms and nullnorms. For a uni-nullnorm $G$ with a 2-neutral element $\{e, 1\}_a$, $G$ is a uninorm if $a = 1$ and a nullnorm if $e = 0$. A uni-nullnorm with a 2-neutral element $\{e, 1\}_a$ is proper if $0 < e < 1$.

Theorem 2.13. (Sun et al. [26]) Let $G : [0, 1]^2 \to [0, 1]$ be a uni-nullnorm such that the underlying t-norms $T^H_G$, $T^ur_G$ and t-conorm $S_G$ are continuous Archimedean. Then $G$ has one of the following structures:

(i) $G(x, y) = \begin{cases} eT^H_G \left( \frac{x}{e}, \frac{y}{e} \right) & \text{if } (x, y) \in [0, e]^2, \\ e + (a - e)S_G \left( \frac{x}{a-e}, \frac{y}{a-e} \right) & \text{if } (x, y) \in [e, a]^2, \\ a + (1 - a)T^ur_G \left( \frac{x-a}{1-a}, \frac{y-a}{1-a} \right) & \text{if } (x, y) \in [a, 1]^2, \\ a & \text{if } (x, y) \in [e, a] \times [a, 1] \cup [a, 1] \times [e, a], \\ \min(x, y) & \text{otherwise.} \end{cases}$

(ii) $G(x, y) = \begin{cases} eT^H_G \left( \frac{x}{e}, \frac{y}{e} \right) & \text{if } (x, y) \in [0, e]^2, \\ e + (a - e)S_G \left( \frac{x}{a-e}, \frac{y}{a-e} \right) & \text{if } (x, y) \in [e, a]^2, \\ a + (1 - a)T^ur_G \left( \frac{x-a}{1-a}, \frac{y-a}{1-a} \right) & \text{if } (x, y) \in [a, 1]^2, \\ a & \text{if } (x, y) \in [0, a] \times [a, 1] \cup [a, 1] \times [0, a], \\ \min(x, y) & \text{otherwise.} \end{cases}$
(iii) 
\[
G(x, y) = \begin{cases} 
  eT_G^l(\frac{x}{e}, \frac{y}{e}) & \text{if } (x, y) \in [0, e]^2, \\
  e + (a - e)S_G(\frac{x-e}{a-e}, \frac{y-e}{a-e}) & \text{if } (x, y) \in [e, a]^2, \\
  a + (1 - a)T_G^{ur}(\frac{x-a}{1-a}, \frac{y-a}{1-a}) & \text{if } (x, y) \in [a, 1]^2, \\
  a & \text{if } (x, y) \in (0, a] \times [a, 1] \cup [a, 1] \times (0, a], \\
  \min(x, y) & \text{otherwise.}
\end{cases}
\]

(iv) 
\[
G(x, y) = \begin{cases} 
  eT_G^l(\frac{x}{e}, \frac{y}{e}) & \text{if } (x, y) \in [0, e]^2, \\
  e + (a - e)S_G(\frac{x-e}{a-e}, \frac{y-e}{a-e}) & \text{if } (x, y) \in [e, a]^2, \\
  a + (1 - a)T_G^{ur}(\frac{x-a}{1-a}, \frac{y-a}{1-a}) & \text{if } (x, y) \in [a, 1]^2, \\
  a & \text{if } (x, y) \in [0, a] \times [a, 1] \cup [a, 1] \times [0, a], \\
  \max(x, y) & \text{otherwise.}
\end{cases}
\]

(v) 
\[
G(x, y) = \begin{cases} 
  eT_G^l(\frac{x}{e}, \frac{y}{e}) & \text{if } (x, y) \in [0, e]^2, \\
  e + (a - e)S_G(\frac{x-e}{a-e}, \frac{y-e}{a-e}) & \text{if } (x, y) \in [e, a]^2, \\
  a + (1 - a)T_G^{ur}(\frac{x-a}{1-a}, \frac{y-a}{1-a}) & \text{if } (x, y) \in [a, 1]^2, \\
  a & \text{if } (x, y) \in [0, a] \times [a, 1] \cup [a, 1] \times [0, a], \\
  0 & \text{if } (x, y) \in \{0\} \times (e, a) \cup (e, a) \times \{0\}, \\
  \max(x, y) & \text{otherwise.}
\end{cases}
\]

(vi) 
\[
G(x, y) = \begin{cases} 
  eT_G^l(\frac{x}{e}, \frac{y}{e}) & \text{if } (x, y) \in [0, e]^2, \\
  e + (a - e)S_G(\frac{x-e}{a-e}, \frac{y-e}{a-e}) & \text{if } (x, y) \in [e, a]^2, \\
  a + (1 - a)T_G^{ur}(\frac{x-a}{1-a}, \frac{y-a}{1-a}) & \text{if } (x, y) \in [a, 1]^2, \\
  a & \text{if } (x, y) \in (0, a] \times [a, 1] \cup [a, 1] \times (0, a], \\
  0 & \text{if } (x, y) \in \{0\} \times [0, 1] \cup [0, 1] \times \{0\}, \\
  \max(x, y) & \text{otherwise.}
\end{cases}
\]

(vii) 
\[
G(x, y) = \begin{cases} 
  aU_G(\frac{x}{a}, \frac{y}{a}) & \text{if } (x, y) \in [0, a]^2, \\
  a + (1 - a)T_G^{ur}(\frac{x-a}{1-a}, \frac{y-a}{1-a}) & \text{if } (x, y) \in [a, 1]^2, \\
  a & \text{if } (x, y) \in (0, a] \times [a, 1] \cup [a, 1] \times (0, a], \\
  0 & \text{otherwise.}
\end{cases}
\]
where $U_G$ is a conjunctive representable uninorm.

\[(viii)\]
\[
G(x, y) = \begin{cases} 
  aU_G\left(\frac{x}{a}, \frac{y}{a}\right) & \text{if } (x, y) \in [0, a]^2, \\
  a + (1 - a)T_G^ur\left(\frac{x - a}{1 - a}, \frac{y - a}{1 - a}\right) & \text{if } (x, y) \in [a, 1]^2, \\
  a & \text{if } (x, y) \in [0, a] \times [a, 1] \cup [a, 1] \times [0, a],
\end{cases}
\]

(10)

where $U_G$ is a disjunctive representable uninorm.

Now, we recall the distributivity equations.

**Definition 2.14.** (Aczél [1]) Let $G, F : [0, 1]^2 \rightarrow [0, 1]$. We say that

(i) $G$ is left distributive over $F$, if for all $x, y, z \in [0, 1]$,

\[G(x, F(y, z)) = F(G(x, y), G(x, z)).\]  

\[(11)\]

(ii) $G$ is right distributive over $F$, if for all $x, y, z \in [0, 1]$,

\[G(F(x, y), z) = F(G(x, z), G(y, z)).\]  

\[(12)\]

If Eqs. (11) and (12) are fulfilled simultaneously, for example, $G$ is commutative, we say that $G$ is distributive over $F$.

**Lemma 2.15.** (Rak [18]) Every increasing operation $F : [0, 1]^2 \rightarrow [0, 1]$ is distributive over max and min.

**Theorem 2.16.** (Ruiz and Torrens [22]) Let $U$ be a representable uninorm with neutral element $f \in (0, 1)$ and $T$ be a continuous t-norm. The following conditions are equivalent.

(i) $U$ is distributive over $T$.

(ii) $U$ is conditionally distributive over $T$.

(iii) We have either one of the following cases:

(a) $T = T_M$.

(b) $T$ is strict and if $t$ is the additive generator of $T$ satisfying $t(f) = 1$, then $\frac{1}{t}$ is also a multiplicative generator of $U$.

**Theorem 2.17.** (Ruiz and Torrens [22]) Let $U$ be a representable uninorm with neutral element $f \in (0, 1)$ and $S$ be a continuous t-conorm. The following conditions are equivalent.

(i) $U$ is distributive over $S$.

(ii) $U$ is conditionally distributive over $S$.

(iii) We have either one of the following cases:

(a) $S = S_M$.

(b) $S$ is strict and if $s$ is the additive generator of $S$ satisfying $s(f) = 1$, then $\frac{1}{s}$ is also a multiplicative generator of $U$. 
3. DISTRIBUTIVITY BETWEEN UNI-NULLNORMS AND CONTINUOUS T-NORMS (T-CONORMS)

In this section, we will discuss distributivity between continuous t-norms (t-conorms) and proper uni-nullnorms with continuous Archimedean underlying t-norms and t-conorms. Here we investigate the distributivity equation between continuous t-norms and uni-nullnorms in detail, and only list the results of this equation for uni-nullnorms and continuous t-conorms as corollaries in this section because t-norms and t-conorms are a pair of dual operators. First of all, let us discuss the distributivity for uni-nullnorms over continuous t-norms.

**Theorem 3.1.** Let uni-nullnorm $G$ be one of Eqs. (3) – (8) with $0 < e < a < 1$ and $T$ be a continuous t-norm. Then $G$ is distributive over $T$ if and only if $T = \min$.

**Proof.** Suppose uni-nullnorm $G$ is one of Eqs. (3), (4), (5). Let $x = a$ and $y = z = 1$ in the distributivity equation, then we easily obtain $a = G(a, 1) = G(a, T(1, 1)) = T(G(a, 1), G(a, 1)) = T(a, a)$.

Now let us prove $T(e, e) = e$. It follows from the continuity of $T$ that there exists $t_1 \in [e, a)$ such that $T(t_1, t_1) = e$. Then we have $x = G(x, e) = G(x, T(t_1, t_1)) = T(G(x, t_1), G(x, t_1)) = T(x \land t_1, x \land t_1) = T(x, x)$ for any $x \in (0, e)$. Thus we obtain $T(e, e) = e$ by the continuity of $T$.

Next we will prove $T$ is idempotent. In fact, we have $x = G(x, e) = G(x, T(e, e)) = T(G(x, e), G(x, e)) = T(x, x)$ for all $x \in [0, a]$, and $x = G(x, 1) = G(x, T(1, 1)) = T(G(x, 1), G(x, 1)) = T(x, x)$ for all $x \in [a, 1]$. That is, $T(x, x) = x$ for all $x \in [0, 1]$. The converse statement is obvious from Lemma 2.15.

Suppose uni-nullnorm $G$ is one of Eqs. (9), (7), (8). Based on the above proof, here we only need to prove $T(e, e) = e$. Assume $T(e, e) < e$, then it follows from the continuity of $T$ that there exists $t_2 \in (e, a)$ such that $T(t_2, t_2) = e$. Then for any $x \in (0, e)$ we have $x = G(x, e) = G(x, T(t_2, t_2)) = T(G(x, t_2), G(x, t_2)) = T(x \lor t_2, x \lor t_2) = T(t_2, t_2) = e$, which is a contradiction. So we have $T(e, e) = e$.

**Theorem 3.2.** Let uni-nullnorm $G$ be one of Eqs. (9), (10) with $0 < e < a < 1$ and $T$ be a continuous t-norm. Then $G$ is distributive over $T$ if and only if one of the following cases holds:

(i) $T = T_M$.

(ii) The structure of $T$ is as follows:

$$T(x, y) = \begin{cases} aT_1(\frac{x}{a}, \frac{y}{a}) & \text{if } (x, y) \in [0, a]^2, \\ \min(x, y) & \text{otherwise,} \end{cases} \quad (13)$$

where $T_1$ is a strict t-norm and if $t_1(\frac{x}{a}) = 1$, then $\frac{1}{t_1}$ is a multiplicative generator of the underlying representable uninorm $U_G$ of uni-nullnorm $G$.

**Proof.** Firstly, it is obvious that $x = G(x, 1) = G(x, T(1, 1)) = T(G(x, 1), G(x, 1)) = T(x, x)$ for all $x \in [a, 1]$. So we have $T(x, y) = \min(x, y)$ for $(x, y) \in [0, a] \times [a, 1] \cup [a, 1] \times$
On the distributivity equation for uni-nullnorms

[0, 1] by the continuity of $T$. Then it follows from Theorem 2.16 that one of the cases (i) and (ii) holds.

Conversely, it is easy to verify that $G$ is distributive over $T$. □

Next we will discuss the distributivity for continuous t-norms over proper uni-nullnorms with continuous Archimedean underlying t-norms and t-conorms.

**Theorem 3.3.** Let $G$ be a proper uni-nullnorm with continuous Archimedean underlying t-norms $T_{ul}^{G}$, $T_{ur}^{G}$ and t-conorm $S_{G}$, and $T$ be a continuous t-norm. Then $T$ is not distributive over $G$.

**Proof.** Let $y = z = 1$ in the distributivity equation, then for any $x \in [0, 1]$ we have $x = T(x, 1) = T(x, G(1, 1)) = G(T(x, 1), T(x, 1)) = G(x, x)$, which contradicts with $G(x, x) < x$ for $x \in (0, e) \cup (a, 1)$ and $G(x, x) > x$ for $x \in (e, a)$. □

**Corollary 3.4.** Let uni-nullnorm $G$ be one of Eqs. (3) – (8) with $0 < e < a < 1$, and $S$ be a continuous t-conorm. Then $G$ is distributive over $S$ if and only if $S = \max$.

**Corollary 3.5.** Let uni-nullnorm $G$ be one of Eqs. (9), (10) with $0 < e < a < 1$ and $S$ be a continuous t-conorm. Then $G$ is distributive over $S$ if and only if one of the following cases holds:

(i) $S = S_{M}$.

(ii) The structure of $S$ is as follows:

$$S(x, y) = \begin{cases} aS_{1}(\frac{x}{a}, \frac{y}{a}) & \text{if } (x, y) \in [0, a]^{2}, \\ \max(x, y) & \text{otherwise,} \end{cases} \tag{14}$$

where $S_{1}$ is a strict t-conorm and if $s_{1}$ is the additive generator of $S_{1}$ satisfying $s_{1}(\frac{e}{a}) = 1$, then $s_{1}$ is also a multiplicative generator of the underlying representable uninorm $U_{G}$ of uni-nullnorm $G$.

**Corollary 3.6.** Let $G$ be a proper uni-nullnorm with continuous Archimedean underlying t-norms $T_{ul}^{G}$, $T_{ur}^{G}$ and t-conorm $S_{G}$, and $S$ be a continuous t-conorm. Then $S$ is not distributive over $G$.

4. DISTRIBUTIVITY BETWEEN UNI-NULLNORMS AND UNINORMS

From now on, $G$ will denote a uni-nullnorm with a 2-neutral element $\{e, 1\}_{a}$, where $0 < e < a < 1$ and the underlying t-norms and t-conorm are continuous Archimedean, and $U \in U_{f}$ will denote a uninorm with a neutral element $0 < f < 1$. Depending on whether the neutral elements of $G$ and $U$ are same or not, there are two cases: distributivity between $G$ and $U$ with $f = e$, and distributivity between $G$ and $U$ with $f \neq e$. 
4.1. Distributivity between $G$ and $U$ with $f = e$

In this section, we will investigate distributivity for $G$ over $U$ and $U$ over $G$ with $f = e$, respectively. Firstly, let us discuss the distributivity equation for uni-nullnorms over uninorms.

**Theorem 4.1.** Let uni-nullnorm $G$ be Eq. [3] with $0 < e < a < 1$ and $U$ be a uninorm with neutral element $e \in (0, 1)$. Then $G$ is distributive over $U$ if and only if the structure of $U$ is as follows:

$$U(x, y) = \begin{cases} 
\max(x, y) & \text{if } (x, y) \in [e, 1]^2, \\
\min(x, y) & \text{otherwise.}
\end{cases}$$ (15)

**Proof.** It is easy for us to prove that $U$ is idempotent. In fact, we have $x = G(x, e) = G(x, U(e, e)) = U(G(x, e), G(x, e)) = U(x, x)$ for any $x \in [0, a]$, and $x = G(x, 1) = G(x, U(1, 1)) = U(G(x, 1), G(1, 1)) = U(x, x)$ for any $x \in [a, 1]$. That is, $U(x, x) = x$ for all $x \in [0, 1]$.

Now let us prove $U(x, y) = \min(x, y)$ for $(x, y) \in [0, e] \times [e, 1]$. Let $(0, e), y \in (0, e)$ and $z \in [e, 1]$, then it follows from the structures of $G$ and $U$ that $G(x, U(y, z)) = U(G(x, y), G(x, z)) = U(y, z)$. Assume there exists some $y_0 \in (0, e)$ and $z_0 \in [e, 1]$ such that $U(y_0, z_0) \geq e$, then for any $x \in (0, e)$ we have $G(x, y_0) = G(x, U(y_0, z_0)) = x \wedge U(y_0, z_0) = x$, which contradicts with the structure of $G$. So we have $U(y, z) \leq e$ for all $(y, z) \in (0, e) \times [e, 1]$. Then from the continuity of the underlying t-norm $T_G^H$ of $G$, it follows that for every $y \in (0, e)$ there must exist $x_0 \in (0, e)$ such that $G(x_0, y) > 0$, that is, $G(x_0, U(y, z)) = G(x_0, y) > 0$. Thus we have $U(y, z) = y = \min(y, z)$ for $(y, z) \in (0, e) \times [e, 1]$ by the conditional cancellation law. From above, we know $U(0, 1) = 0$ by the monotonicity of $U$, so $U(0, z) = 0$ for $z \in [e, 1]$. Therefore, $U(y, z) = y = \min(y, z)$ for $(y, z) \in [0, e] \times [e, 1]$.

The converse statement is obvious.

**Theorem 4.2.** Let uni-nullnorm $G$ be Eq. [4] with $0 < e < a < 1$ and $U$ be a uninorm with neutral element $e \in (0, 1)$. Then $G$ is distributive over $U$ if and only if the structure of $U$ is as follows:

$$U(x, y) = \begin{cases} 
\max(x, y) & \text{if } (x, y) \in [0, e] \times [a, 1] \cup [a, 1] \times [0, e] \cup [e, 1]^2, \\
\min(x, y) & \text{otherwise.}
\end{cases}$$ (16)

**Proof.** Based on the proof of Theorem 4.1, we only need to prove $U(x, y) = \max(x, y)$ for $(x, y) \in [0, e] \times [a, 1]$. Firstly, let $x \in [0, e]$, $y = e$ and $z \in [a, 1]$ in distributivity equation, then $a = G(x, z) = G(x, U(e, z)) = U(G(x, e), G(x, z)) = U(a, x)$ for any $x \in [0, e]$.

Now let $x \in (a, 1]$, $y \in [0, e]$ and $z \in (a, 1]$, then it follows from the structures of $G$ and $U$ that $G(x, U(y, z)) = U(G(x, y), G(x, z)) = U(a, G(x, z)) = a \lor G(x, z) = G(x, z)$. So we have $U(y, z) = z = \max(y, z)$ for $(y, z) \in [0, e] \times (a, 1]$ since the underlying operator $T_G^ur$ of $G$ is a continuous Archimedean t-norm, which satisfies the conditional cancellation law.

Conversely, it is easy to verify that $G$ is distributive over $U$. 

□
Theorem 4.3. Let uni-nullnorm $G$ be Eq.(5) with $0 < e < a < 1$ and $U$ be a uninorm with neutral element $e \in (0, 1)$. Then $G$ is distributive over $U$ if and only if the structure of $U$ is as follows:

$$U(x, y) = \begin{cases} \max(x, y) & \text{if } (x, y) \in (0, e) \times [a, 1] \cup [a, 1] \times (0, e) \cup [e, 1]^2, \\ 0 & \text{if } (x, y) \in \{0\} \times [a, 1] \cup [a, 1] \times \{0\}, \\ \min(x, y) & \text{otherwise.} \end{cases}$$ (17)

Proof. Based on the proof of Theorem 4.2, here we only prove $U(0, a) = U(0, 1) = 0$. Firstly, take $x = a, y = 0$ and $z = e$ in distributivity equation, then we have $0 = G(a, 0) = G(a, U(0, e)) = U(G(a, 0), G(a, e)) = U(0, a)$. And take $x = e, y = 0$ and $z = 1$ in distributivity equation, then we have $G(e, U(0, 1)) = U(G(e, 0), G(e, 1)) = U(0, a) = 0$. Suppose $U(0, 1) = 1$, then it follows from the structure of $G$ that $0 = G(e, U(0, 1)) = G(e, 1) = a$, which is a contradiction. So we have $U(0, 1) = 0$. □

![Fig. 1. (Left:) $U$ in Theorem 4.1 (middle:) $U$ in Theorem 4.2 (right:) $U$ in Theorem 4.3](image)

Theorem 4.4. Let uni-nullnorm $G$ be Eq.(6) with $0 < e < a < 1$ and $U$ be a uninorm with neutral element $e \in (0, 1)$. Then $G$ is distributive over $U$ if and only if the structure of $U$ is as follows:

$$U(x, y) = \begin{cases} \min(x, y) & \text{if } (x, y) \in [0, e]^2, \\ \max(x, y) & \text{otherwise.} \end{cases}$$ (18)

Proof. Firstly, we know that $U$ is idempotent from the proof of Theorem 4.1. Let $x \in [0, e], y = e$ and $z \in [a, 1]$ in distributivity equation, then we have $a = G(x, z) = G(x, U(e, z)) = U(G(x, e), G(x, z)) = U(x, a)$, That is, $U(x, a) = a$ for all $x \in [0, e]$.

Now let us prove $U(x, y) = \max(x, y)$ for $(x, y) \in [0, e] \times (e, 1)$. Let $x \in (e, a), y \in [0, e]$ and $z \in (e, a)$, then it follows from the structures of $G$ and $U$ that $G(x, U(y, z)) = U(G(x, y), G(x, z)) = U(x, G(y, z)) = x \lor G(x, z) = G(x, z)$. Assume there exists some $y' \in [0, e]$ and $z' \in (e, a)$ such that $U(y', z') \leq e$, then for any $x \in (e, a)$ we have $x = x \lor U(y', z') = G(x, U(y', z')) = G(x, z')$, which contradicts with the structure of $G$. So $e < U(y, z) \leq a$ for all $(y, z) \in [0, e] \times (e, a)$. Thus we obtain
$U(y, z) = z = \max(y, z)$ for $(y, z) \in [0, e] \times (e, a)$ since the underlying operator $S_G$ is a continuous Archimedean t-conorm, which satisfies the conditional cancellation law. And let $x \in (a, 1]$, $y \in [0, e]$ and $z \in (a, 1]$, it follows from the structures of $G$ and $U$ that $G(x, U(y, z)) = U(G(x, y), G(x, z)) = U(a, G(x, z)) = a \lor G(x, z) = G(x, z)$, then $U(y, z) = z = \max(y, z)$ for $(y, z) \in [0, e] \times (a, 1]$ since the underlying operator $T^U_{a^+}$ is a continuous Archimedean t-norm, which satisfies the conditional cancellation law.

The converse statement is obvious. □

**Theorem 4.5.** Let uni-nullnorm $G$ be Eq.(7) with $0 < e < a < 1$ and $U$ be a uninorm with neutral element $e \in (0, 1)$. Then $G$ is distributive over $U$ if and only if the structure of $U$ is as follows:

$$U(x, y) = \begin{cases} 
\min(x, y) & \text{if } (x, y) \in [0, e]^2, \\
0 & \text{if } (x, y) \in \{0\} \times (e, a) \cup (e, a) \times \{0\}, \\
\max(x, y) & \text{otherwise.}
\end{cases}$$ (19)

**Proof.** Based on the proof of Theorem 4.4, we just prove $U(x, 0) = 0$ for $x \in (e, a)$. Let $x \in (e, a)$, $y \in (0, e]$ and $z = 0$ in distributivity equation, then we have $0 = G(x, 0) = G(x, U(y, 0)) = U(G(x, y), G(x, 0)) = U(x \lor y, 0) = U(x, 0)$. □

**Theorem 4.6.** Let uni-nullnorm $G$ be Eq.(8) with $0 < e < a < 1$ and $U$ be a uninorm with neutral element $e \in (0, 1)$. Then $G$ is distributive over $U$ if and only if the structure of $U$ is as follows:

$$U(x, y) = \begin{cases} 
\min(x, y) & \text{if } (x, y) \in [0, e]^2, \\
0 & \text{if } (x, y) \in \{0\} \times [e, 1] \cup [e, 1] \times \{0\}, \\
\max(x, y) & \text{otherwise.}
\end{cases}$$ (20)

**Proof.** Based on the proof of Theorem 4.5, we just prove $U(0, a) = U(0, 1) = 0$. Take $x = a$, $y = 0$ and $z \in [0, e]$ in distributivity equation, we have $0 = G(a, 0) = G(a, U(0, z)) = U(G(a, 0), G(a, z)) = U(0, a)$. And take $x = a$, $y = 0$ and $z = 1$ in distributivity equation, we have $G(a, U(0, 1)) = U(G(a, 0), G(a, 1)) = U(0, a) = 0$. Suppose $U(0, 1) = 1$, then $a = G(a, 1) = G(a, U(0, 1)) = 0$, which is a contradiction. So we have $U(0, 1) = 0$. □

The distributivity for representable uninorms over uninorms was already studied in [21], and it was proved that there are no solutions in this case. That is, representable uninorms are not distributive over uninorms. Thus, we have the following conclusion.

**Theorem 4.7.** Let uni-nullnorm $G$ be one of Eqs. [9], [10] with $0 < e < a < 1$ and $U$ be a uninorm with neutral element $e \in (0, 1)$. Then $G$ is not distributive over $U$.

**Proof.** First of all, we know that $U$ is idempotent from the proof of Theorem 4.1. Let $x \in (0, e)$, $y \in (e, a)$ and $z = e$ in distributivity equation, then we have $G(x, y) = G(x, U(y, e)) = U(G(x, y), G(x, e)) = U(G(x, y), x)$. 
Now let us prove $U(x, y) \neq e$ for $(x, y) \in (0, e) \times (e, a)$. Assume there exists some $x_0 \in (0, e)$ and $y_0 \in (e, a)$ such that $G(x_0, y_0) = e$, then $e = G(x_0, y_0) = U(G(x_0, y_0), x_0) = U(e, x_0) = x_0$, which is a contradiction. So we have $G(x, y) \neq e$ and $x = G(x, e) < G(x, y) < G(e, y) = y$ for $(x, y) \in (0, e) \times (e, a)$ from the strict monotonicity of the underlying representable uninorm $U_G$ of $G$. Thus, there are two possibilities: $0 < x < G(x, y) < e < y < a$ and $0 < x < e < G(x, y) < y < a$. When $0 < x < G(x, y) < e < y < a$, then from the structure of $U$ it follows that $G(x, y) = U(G(x, y), x) = G(x, y) \wedge x = x$, which is a contradiction. When $0 < x < e < G(x, y) < y < a$, then from the structure of $U$ it follows that $G(x, y) = G(y, x) = U(G(y, x), y) = G(x, y) \vee y = y$, which is a contradiction. Therefore, $G$ is not distributive over $U$.

Next we will discuss the distributivity for uninorms over proper uni-nullnorms with continuous Archimedean underlying t-norms and t-conorms.

**Theorem 4.8.** Let $G$ be a proper uni-nullnorm with continuous Archimedean underlying t-norms $T^l_G$, $T^r_G$ and t-conorm $S_G$, and $U$ be a uninorm with neutral element $e \in (0, 1)$. Then $U$ is not distributive over $G$.

**Proof.** The proof is similar to the one of Theorem 3.3.

**4.2. Distributivity between $G$ and $U$ with $f \neq e$**

In this section, we will investigate distributivity for $G$ over $U$ and $U$ over $G$ with $f \neq e$, respectively. Firstly, let us discuss the distributivity equation for uni-nullnorms over uninorms.

**Theorem 4.9.** Let uni-nullnorm $G$ be Eq. (3) with $0 < e < a < 1$ and $U$ be a uninorm with neutral element $f \neq e$, where the underlying t-norm $T^l_U$ is continuous. Then $G$ is distributive over $U$ if and only if $f = a$ and $U$ has the form $U^{[5]}$.

**Proof.** First of all, let us prove $f = a$. Assume $f \in (0, e)$, then we know $G(f, f) < f$ from Lemma 2.3. Taking $x = z = f$ and $y = e$ in distributivity equation, then we have
$f = G(f, e) = G(f, U(e, f)) = U(G(f, e), G(f, f)) = U(f, G(f, f)) = G(f, f)$, which is a contradiction. Assume $f \in (e, a)$, then we know $G(f, f) > f$. Taking $x = z = f$ and $y = e$ in distributivity equation, then we have $f = G(f, e) = G(f, U(e, f)) = U(G(f, e), G(f, f)) = U(f, G(f, f)) = G(f, f)$, which is a contradiction. Assume $f \in (a, 1)$, then we know $G(f, f) < f$ from Lemma 2.3. Taking $x = z = f$ and $y = 1$ in distributivity equation, then we have $f = G(f, 1) = G(f, U(1, f)) = U(G(f, 1), G(f, f)) = U(f, G(f, f)) = G(f, f)$, which is a contradiction. So we have $f = a$.

Next, let us prove that $U$ is idempotent. From the structure of $U$, it follows that $U(e, e) \leq e$, then there must exist $t_1 \in (e, a)$ such that $U(t_1, t_1) = e$ by the continuity of $T_U$. So for any $x \in (0, e)$, we have $x = G(x, e) = G(x, U(t_1, t_1)) = U(G(x, t_1), G(x, t_1)) = U(x \wedge t_1, x \wedge t_1) = U(x, x)$. Thus, we obtain $U(e, e) = e$ by the continuity of $T_U$. Therefore, we have $x = G(x, e) = G(x, U(e, e)) = U(G(x, e), G(x, e)) = U(x, x)$ for any $x \in [0, a]$ and $x = G(x, 1) = G(x, U(1, 1)) = U(G(x, 1), G(x, 1)) = U(x, x)$ for any $x \in [a, 1]$. That is, $U$ is idempotent. And similar to the proof of Theorem 4.1, we have $U(x, y) = \min(x, y)$ for $(x, y) \in [0, a] \times [a, 1] \cup [a, 1] \times [0, a]$. The converse statement is obvious from Lemma 2.15. 

**Theorem 4.10.** Let uni-nullnorm $G$ be one of Eqs. (4) – (10) with $0 < e < a < 1$ and $U$ be a uninorm with neutral element $f \neq e$. Then $G$ is distributive over $U$ if and only if $f = a$ and $U$ has the form (15).

**Proof.** The proof is similar to the one of Theorem 4.9. 

**Theorem 4.11.** Let uni-nullnorm $G$ be Eq. (3) with $0 < e < a < 1$ and $U$ be a uninorm with neutral element $f \neq e$. Then $U$ is not distributive over $G$.

**Proof.** Suppose $U$ is distributive over $G$. If $f \in (0, e)$, then we have $x = U(x, f) = U(x, G(e, f)) = G(U(x, e), U(x, f)) = G(U(x, e), x)$ for $x \in (e, a)$. Assume there exists $x_0 \in (e, a)$ such that $U(x_0, e) \geq a$, then from the structure of $G$ we have $x_0 = G(U(x_0, e), x_0) = a$, which is a contradiction. That is, $e \leq U(x, e) < a$ for $x \in (e, a)$. Thus, from the structure of the underlying t-conorm $S_G$ of $G$, it follows that $G(x, x) > x = G(U(x, e), x)$ for any $x \in (e, a)$. So we obtain $U(x, e) < x$ for $x \in (e, a)$, which contradicts with the fact that $U(x, e) \geq x$ for $x \in (e, a)$ by the structure of $U$. If $f \in (a, 1)$, then we have $x = U(x, f) = U(x, G(e, f)) = G(U(x, e), U(x, f)) = G(U(x, e), x)$ for $x \in (0, e)$. From the structure of the underlying t-norm $T^n_U$ of $G$, it follows that $G(x, x) < x = G(U(x, e), x)$ for any $x \in (0, e)$. So we obtain $U(x, e) > x$ for $x \in (0, e)$, which contradicts with the fact that $U(x, e) \leq x$ for $x \in (0, e)$ by the structure of $U$. If $f \in (a, 1)$, then we have $x = U(x, f) = U(x, G(1, f)) = G(U(x, 1), U(x, f)) = G(U(x, 1), x)$ for $x \in (e, a)$. Assume there exists $x_0 \in (e, a)$ such that $U(x_0, 1) \geq a$, then from the structure of $G$ we have $x_0 = G(U(x_0, 1), x_0) = a$, which is a contradiction. That is, $e < x \leq U(x, 1) < a$ for $x \in (e, a)$. Thus, from the structure of the underlying t-conorm $S_G$ of $G$, it follows that $G(x, x) > x = G(U(x, 1), x)$ for any $x \in (e, a)$. So we obtain $U(x, 1) < x$ for $x \in (e, a)$, which contradicts with the fact that $U(x, 1) \geq x$ for $x \in (e, a)$ by the structure of $U$. 

**Theorem 4.12.** Let uni-nullnorm $G$ be one of Eqs. (1) – (8) with $0 < e < a < 1$ and $U$ be a uninorm with neutral element $f \neq e$. Then $U$ is not distributive over $G$. 
Proof. The proof is similar to the one of Theorem 4.11.

Theorem 4.13. Let uni-nullnorm $G$ be Eq.(9) with $0 < e < a < 1$ and $U$ be a uninorm with neutral element $f \neq e$. Then $U$ is not distributive over $G$.

Proof. Suppose $U$ is distributive over $G$. If $f \in (0, e)$, then we have $x = U(x, f) = U(x, G(e, f)) = G(U(x, e), U(x, f)) = G(U(x, e), x)$ for $x \in (e, a)$. Assume there exists $x_0 \in (0, a)$ such that $U(x_0, e) \geq a$, then by the structure of $G$ we have $x_0 = G(U(x_0, e), x_0) = a$, which is a contradiction. That is, $U(x, e) < a$ for $x \in (0, a)$. Thus, from the structure of the underlying representable uninorm $U_G$ of $G$, it follows that $G(x, e) = x = G(U(x, e), x)$ for any $x \in (0, a)$. So we obtain $U(x, e) = e$ for $x \in (0, a)$, which contradicts with the fact that $U(x, e) \geq x > e$. And we can similarly obtain contradictions if $f \in (e, 1)$.

Theorem 4.14. Let uni-nullnorm $G$ be Eq.(10) with $0 < e < a < 1$ and $U$ be a uninorm with neutral element $f \neq e$. Then $U$ is not distributive over $G$.

Proof. The proof is similar to the one of Theorem 4.13.

5. DISTRIBUTIVITY BETWEEN UNI-NULLNORMS AND NULLNORMS

From now on, $V \in V_k$ will denote a nullnorm with a zero element $0 < k < 1$. Depending on whether the zero elements of $G$ and $V$ are same or not, there are two cases: distributivity between $G$ and $V$ with $k = a$, and distributivity between $G$ and $V$ with $k \neq a$.

5.1. Distributivity between $G$ and $V$ with $k = a$

In this section, we will investigate distributivity for $G$ over $V$ and $V$ over $G$ with $k = a$, respectively. Firstly, let us discuss the distributivity equation for uni-nullnorms over nullnorms.

Theorem 5.1. Let uni-nullnorm $G$ be Eq.(3) with $0 < e < a < 1$ and $V$ be a nullnorm with zero element $a \in (0, 1)$. If $G$ is distributive over $V$, then $V(x, y) = \max(x, y)$ for $(x, y) \in [0, e]^2$ and $V(x, y) = \min(x, y)$ for $(x, y) \in [a, 1]^2$.

Proof. Taking $y = z = e$ and $x \in [0, e]$ in distributivity equation, then we have $x = x \land V(e, e) = G(x, V(e, e)) = V(G(x, e), G(x, e)) = V(x, x)$. And let $y = z = 1$ and $x \in [a, 1]$ in distributivity equation, we have $x = G(x, 1) = G(x, V(1, 1)) = V(G(x, 1), G(x, 1)) = V(x, x)$. That is, $V(x, y) = \max(x, y)$ for $(x, y) \in [0, e]^2$ and $V(x, y) = \min(x, y)$ for $(x, y) \in [a, 1]^2$.

Theorem 5.2. Let uni-nullnorm $G$ be Eq.(4) with $0 < e < a < 1$ and $V$ be a nullnorm with zero element $a \in (0, 1)$. If $G$ is distributive over $V$, then $e \leq V(e, e) < a$, and $V(x, y) = \max(x, y)$ for $(x, y) \in [0, e]^2$ and $V(x, y) = \min(x, y)$ for $(x, y) \in [a, 1]^2$. Especially, if $V(e, e) = e$, then $G$ is distributive over $V$ if and only if $V$ is idempotent.
Proof. Based on the proof of Theorem 5.1 we only need to prove \( V(e, e) < a \). Assume \( V(e, e) = a \), then we have \( a = G(0, a) = G(0, V(e, e)) = V(G(0, e), G(0, e)) = V(0, 0) = 0 \), which is a contradiction.

Theorem 5.3. Let uni-nullnorm \( G \) be Eq. (5) with \( 0 < e < a < 1 \) and \( V \) be a nullnorm with zero element \( a \in (0, 1) \). Then one of the following cases holds:

(i) If \( V(e, e) = e \), then \( G \) is distributive over \( V \) if and only if \( G \) is idempotent.

(ii) If \( e < V(e, e) < a \), and \( G \) is distributive over \( V \), then \( V(x, y) = \max(x, y) \) for \( (x, y) \in [0, e)^2 \) and \( V(x, y) = \min(x, y) \) for \( (x, y) \in [a, 1]^2 \).

(iii) If \( V(e, e) = a \), then \( G \) is distributive over \( V \) if and only if the underlying t-norm \( T^l_G \) is strict and the structure of \( V \) is as follows:

\[
V(x, y) = \begin{cases} 
\min(x, y) & \text{if } (x, y) \in [a, 1]^2, \\
y & \text{if } (x, y) \in \{0\} \times [0, a], \\
x & \text{if } (x, y) \in [0, a] \times \{0\}, \\
a & \text{otherwise.}
\end{cases}
\] (21)

Proof. We only prove (iii) here. For any \( x \in (0, a] \), we can easily obtain \( a = G(x, a) = G(x, V(e, e)) = V(G(x, e), G(x, e)) = V(x, x) \). So for any \( x, y \in (0, a] \), we have \( a = V(x \land y, x \land y) \leq V(x, y) \leq V(x \lor y, x \lor y) = a \). Now assume the underlying t-norm \( T^l_G \) is nilpotent, then there must exist \( t \in (0, e) \) such that \( G(t, t) = 0 \). Thus, for any \( x, y, z \in (0, t] \), we have \( a = G(x, a) = G(x, V(y, z)) = V(G(x, y), G(x, z)) = V(0, 0) = 0 \), which is a contradiction.

Conversely, in order to verify that \( G \) is distributive over \( V \), we need to consider the following cases by the commutativity of \( V \).

1) When \( y, z \in (0, a] \).
   If \( x > 0 \), then \( G(x, V(y, z)) = G(x, a) = a = V(G(x, y), G(x, z)) \) by the structures of \( G \) and \( V \).
   If \( x = 0 \), then \( G(x, V(y, z)) = G(0, a) = 0 = V(0, 0) = V(G(0, y), G(0, z)) = V(G(x, y), G(x, z)) \) by the structures of \( G \) and \( V \).

2) When \( y = 0 \) and \( z \in [0, a] \). It is obvious for any \( x \in [0, 1] \) that \( G(x, V(0, z)) = G(x, z) = V(0, G(x, z)) = V(G(x, 0), G(x, z)) \) by the structures of \( G \) and \( V \).

3) When \( y \in [0, a] \) and \( z \in [a, 1] \).
   If \( x > 0 \), then \( G(x, V(y, z)) = G(x, a) = a = V(G(x, y), G(x, z)) \) by the structures of \( G \) and \( V \).
   If \( x = 0 \), then \( G(x, V(y, z)) = G(0, a) = 0 = V(0, 0) = V(G(0, y), G(0, z)) = V(G(x, y), G(x, z)) \) by the structures of \( G \) and \( V \).

4) When \( y, z \in [a, 1] \). Without loss of generality, we assume \( y \leq z \).
   If \( x > 0 \), then \( G(x, V(y, z)) = G(x, y \land z) = G(x, y) \land G(x, z) = V(G(x, y), G(x, z)) \) by the structures of \( G \) and \( V \).
   If \( x = 0 \), then \( G(x, V(y, z)) = G(0, y) = 0 = V(0, 0) = V(G(0, y), G(0, z)) = V(G(x, y), G(x, z)) \) by the structures of \( G \) and \( V \).
**Theorem 5.4.** Let uni-nullnorm $G$ be Eq.(6) with $0 < e < a < 1$ and $V$ be a nullnorm with zero element $a \in (0, 1)$. Then $G$ is distributive over $V$ if and only if $V$ is idempotent.

**Proof.** In order to prove that $V$ is idempotent, we just need to prove $V(e, e) = e$. Firstly, we know that $e \leq V(e, e) \leq a$ by the structure of $V$. Assume $V(e, e) = a$, then we have $a = G(0, a) = G(0, V(e, e)) = V(G(0, e), G(0, e)) = V(0, 0) = 0$, which is a contradiction. Assume $e < V(e, e) < a$, then we have $V(e, e) = V(e, e) = G(0, V(e, e)) = V(G(0, e), G(0, e)) = V(0, 0) = 0$, which is a contradiction. So $V(e, e) = e$.

**Theorem 5.5.** Let uni-nullnorm $G$ be Eq.(7) with $0 < e < a < 1$ and $V$ be a nullnorm with zero element $a \in (0, 1)$. If $G$ is distributive over $V$, then $e \leq V(e, e) < a$. Especially,

(i) If $V(e, e) = e$, then $G$ is distributive over $V$ if and only if $V$ is idempotent.

(ii) If $e < V(e, e) < a$, and $G$ is distributive over $V$, then $V(x, y) = V(e, e)$ for $(x, y) \in (0, e]^2$ and $V(x, y) = \min(x, y)$ for $(x, y) \in [a, 1]^2$.

**Proof.** The proof is similar to the one of Theorem 5.1 and Theorem 5.2.

**Theorem 5.6.** Let uni-nullnorm $G$ be Eq.(8) with $0 < e < a < 1$ and $V$ be a nullnorm with zero element $a \in (0, 1)$. Then one of the following cases holds:

(i) If $V(e, e) = e$, then $G$ is distributive over $V$ if and only if $V$ is idempotent.

(ii) If $e < V(e, e) < a$, and $G$ is distributive over $V$, then $V(x, y) = V(e, e)$ for $(x, y) \in (0, e]^2$ and $V(x, y) = \min(x, y)$ for $(x, y) \in [a, 1]^2$.

(iii) If $V(e, e) = a$, then $G$ is distributive over $V$ if and only if the underlying t-norm $T_G^l$ is strict and the structure of $V$ is Eq. (21).

**Proof.** The proof is similar to the one of Theorem 5.3.

**Theorem 5.7.** Let uni-nullnorm $G$ be one of Eqs. (9), (10) with $0 < e < a < 1$ and $V$ be a nullnorm with zero element $a \in (0, 1)$. If $G$ is distributive over $V$, then $V(x, y) = \min(x, y)$ for $(x, y) \in [a, 1]^2$. Especially, if the underlying t-conorm $S_V$ of $V$ is continuous, then $G$ is distributive over $V$ if and only if one of the following cases holds:

(i) $V$ is idempotent.

(ii) The structure of $V$ is as follows:

$$V(x, y) = \begin{cases} \ aS_V(\frac{x}{a}, \frac{y}{a}) & \text{if } (x, y) \in [0, a]^2, \\ \min(x, y) & \text{if } (x, y) \in [a, 1]^2, \\ \ a & \text{otherwise,} \end{cases} \quad (22)$$

where the underlying t-conorm $S_V$ of $V$ is strict, and if $s$ is the additive generator of $S_V$ satisfying $s(\frac{x}{a}) = 1$, is also a multiplicative generator of $U_G$. 


Proof. The proof is similar to the one of Theorem 3.2.

Next we will discuss the distributivity for nullnorms over proper uni-nullnorms with continuous Archimedean underlying t-norms and t-conorms.

Theorem 5.8. Let $G$ be a proper uni-nullnorm with continuous Archimedean underlying t-norms $T_{G}^{ll}$, $T_{G}^{ur}$ and t-conorm $S_{G}$, and $V$ be a nullnorm. Then $V$ is not distributive over $G$.

Proof. The proof is similar to the one of Theorem 3.3.

5.2. Distributivity between $G$ and $V$ with $k \neq a$

In this section, we will investigate distributivity for $G$ over $V$ and $V$ over $G$ with $k \neq a$, respectively. Firstly, let us discuss the distributivity equation for uni-nullnorms over nullnorms.

Theorem 5.9. Let uni-nullnorm $G$ be Eq.(3) with $0 < e < a < 1$ and $V$ be a nullnorm with zero element $k \neq a$. Then $G$ is not distributive over $V$.

Proof. Suppose $G$ is distributive over $V$.

If $0 < k < e$, then we have $G(k, k) = G(k, V(0, e)) = V(G(k, 0), G(k, e)) = V(0, k) = k$, which contradicts with $G(x, x) < x$ for all $x \in (0, e)$.

If $k = e$, then we have $a = G(1, e) = G(1, V(1, 0)) = V(G(1, 1), G(1, 0)) = V(1, 0) = e$, which is a contradiction.

If $e < k < a$, then we have $G(k, k) = G(k, V(a, e)) = V(G(k, a), G(k, e)) = V(a, k) = k$, which contradicts with $G(x, x) > x$ for all $x \in (e, a)$.

If $a < k < 1$, then we have $G(k, k) = G(k, V(a, 1)) = V(G(k, a), G(k, 1)) = V(a, k) = k$, which contradicts with $G(x, x) < x$ for all $x \in (a, 1)$.

Theorem 5.10. Let uni-nullnorm $G$ be one of Eqs. (4) – (10) with $0 < e < a < 1$ and $V$ be a nullnorm with zero element $k \neq a$. Then $G$ is not distributive over $V$. 
Proof. The proof is similar to the one of Theorem 5.9.

Theorem 5.11. Let $G$ be a proper uni-nullnorm with continuous Archimedean underlying t-norms $T^l_G$, $T^u_G$ and t-conorm $S_G$, and $V$ be a nullnorm with $k \neq a$. Then $V$ is not distributive over $G$.

Proof. Suppose that $V$ is distributive over $G$, then we have $x = V(x, 1) = V(x, G(1, 1)) = G(V(x, 1), V(x, 1)) = G(x, x)$ for all $x \in (k, 1)$, which contradicts with the structure of the underlying t-norm $T^u_G$ of $G$.

From the above Theorems, we can easily obtain the following conclusion.

Remark. Let $G$ and $F$ be two proper uni-nullnorms with continuous Archimedean underlying t-norms and t-conorm. Then $G$ is not distributive over $F$.

6. CONCLUSION AND FURTHER WORK

In this paper, we investigated the distributivity between uni-nullnorms and some other binary operators, such as, continuous t-norms, continuous t-conorms, uninorms and nullnorms. From the above Theorems and corollaries, we know that there is no solution of distributivity equation for continuous t-norms (continuous t-conorms, uninorms or nullnorms) over uni-nullnorms, which have continuous Archimedean underlying t-norms and t-conorms. As for the distributivity for uni-nullnorms with continuous Archimedean underlying t-norms and t-conorms over continuous t-norms, continuous t-conorms or uninorms, we have obtained full characterizations. But for the distributivity for uni-nullnorms over nullnorms, we obtained partial characterizations about nullnorms.

In the future work, we will consider characterizations of uni-nullnorms with continuous or left-continuous underlying t-norms and t-conorms. And we also want to investigate other functional equations about uni-nullnorms.

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Ya-Ming Wang, School of Mathematics, Shandong University, Jinan, 250100. P. R. China.
e-mail: 623073044@qq.com

Hua-Wen Liu, School of Mathematics, Shandong University, Jinan, 250100. P. R. China.
e-mail: hw.liu@sdu.edu.cn